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Kowalevski's analysis of the swinging Atwood's machine

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Abstract
We study the Kowalevski expansions near singularities of the swinging Atwood's machine. We show that there is an infinite number of mass ratios M/m where such expansions exist with the maximal number of arbitrary constants. These expansions are of the so-called weak Painlevé type. However, in view of these expansions, it is not possible to distinguish between integrable and nonintegrable cases.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The swinging Atwood's machine is a variable length pendulum of mass m on the left, and a nonswinging mass M on the right, tied together by a string, in a constant gravitational field, see figure 1. The coupling of the two masses is expressed by the fact that the length of the string is fixed:

$$\sqrt{x^2+y^2}+|z|=L, \qquad \Longrightarrow x^2+y^2=(|z|-L)^2.$$

Up to a choice of origin for z, one can assume $L=0$, so the constraint is the cone $z^2=x^2+y^2$. To describe the dynamics we choose to work with constrained variables and write a Lagrangian

$$\mathcal{L}=\frac{m}{2}(\dot{x}^2+\dot{y}^2)+\frac{M}{2}\dot{z}^2-g(my+Mz)+\frac{\lambda}{2}(x^2+y^2-z^2),$$

where λ, a Lagrange multiplier (of dimension MT^{-2}), has been introduced, whose equation of motion enforces the constraint. The equations of motion read

$$m\ddot{x}=\lambda x \tag{1}$$

$$m\ddot{y}=-mg+\lambda y \tag{2}$$

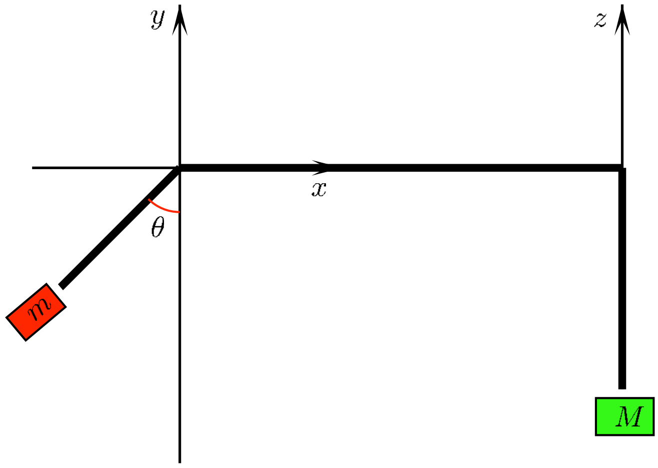

Figure 1. Swinging Atwood's machine.

$$M\ddot{z} = -Mg - \lambda z \tag{3}$$

$$0 = x^2 + y^2 - z^2. \tag{4}$$

From these equations, one can express λ in terms of positions and velocities:

$$\lambda = \frac{x\ddot{x} + y\ddot{y} - z\ddot{z} + g(y - z)}{\frac{1}{m}(x^2 + y^2) + \frac{1}{M}z^2} = \frac{mM}{M + m}\frac{\dot{z}^2 - \dot{x}^2 - \dot{y}^2 + g(y - z)}{z^2}. \tag{5}$$

Alternatively, rescaling

$$x \to \frac{1}{\sqrt{m}}x, \qquad y \to \frac{1}{\sqrt{m}}y, \qquad z \to \frac{1}{\sqrt{M}}z$$

we can view the system as a unit mass particle moving on a cone

$$z^2 = \frac{M}{m}(x^2 + y^2)$$

subjected to a constant field force

$$\begin{pmatrix} f_x \\ f_y \\ fz \end{pmatrix} = \begin{pmatrix} 0 \\ -g\sqrt{m} \\ -g\sqrt{M} \end{pmatrix}.$$

The slope of the force in the (y, z) plane coincides with the angle of the cone.

The swinging Atwood's machine has been studied in great detail by Tufillaro and his co-workers, see [1–7]. They have first studied numerically the equations of motion and shown that for most values of the mass ratio M/m the motion appears to be chaotic; however, for some values, like 3, 15, etc the motion seems less chaotic and could perhaps be integrable. In a further study, Tufillaro [4] showed that the system is indeed integrable for $M/m = 3$ by exhibiting a change of coordinates, somewhat related to parabolic coordinates, in which separation of variables occurs. He was then able to solve the equations of motion in terms of elliptic functions, which is quite peculiar since in general integrable systems with two degrees of freedom can be solved only in terms of hyperelliptic functions, such as for the Kowalevski top [8]. He also obtained the second conserved quantity which ensures integrability. In the same paper, he conjectured that the system is integrable for $M/m = 15, \ldots, 4n^2 - 1$, with n integer.

However, later on, Casasayas *et al* proved [6] that the system can be integrable for discrete values of the ratio M/m only in the interval $]1, 3]$, using nonintegrability theorems developed by Yoshida [9] and Ziglin. The essence of the Yoshida–Ziglin argument is to study the monodromy developed by Jacobi variations around an exact solution, when the time variable describes a loop in the complex plane. The monodromy must preserve conserved quantities, but this is impossible in general if the monodromy group is not abelian. In the case at hand, one can compute monodromies from hypergeometric equations and conclude. We have also been informed by private communication from J P Ramis that he and his co-workers have proven that the swinging Atwood's machine is never integrable except for $M/m = 3$, using methods from differential Galois theory.

The aim of our paper is to work out the Kowalevski analysis for this model. Let us recall the idea of the Kowalevski method. If a dynamical system is *algebraically* integrable one can expect to obtain expressions for the dynamical variables in terms of quotients of theta functions defined on the Jacobian of some algebraic curve of genus g, where $g = 2$ for a system with two degrees of freedom. Only quotients may appear because theta functions have monodromy on the Jacobian torus, which needs to be canceled. Hence, denominators which can vanish for any given initial conditions and for some finite value, in general complex, of time will appear in the solution. Hence, the equations of motion must admit Laurent solutions—that is divergent for some value of time, with as many parameters as there are initial conditions. Kowalevski first noted [8] that this imposes strong constraints on these equations, from which she was able to deduce the celebrated Kowalevski case of the top equation.

Looking for Laurent solutions to the swinging Atwood's machine equations of motion in the integrable case $M/m = 3$ we first noted that there are none, but there exist the so-called weak Painlevé solutions, that is Laurent developments not in the time variable t but in some radical $t^{1/k}$, generally called Puiseux expansions.

It had already been discovered by Ramani *et al* [10] that some integrable systems require weakening the Kowalevski–Painlevé analysis to obtain expansions at infinity of dynamical variables. This may be explained in general, and is certainly the case for our example, by the fact that there is a 'better' variable which has true Laurent expansions and time itself can be expressed in terms of this variable through an algebraic equation which happens to produce the given radicals. Moreover Ramani *et al* advocated the idea that the existence of weak Painlevé solutions is a criterion of integrability, like in the Kowalevski's case.

For our model of the swinging Atwood's machine, we find that there are weak Painlevé solutions not only when $M/m = 15$ but for a whole host of other values of the mass ratio, all of them corresponding to obviously nonintegrable cases. Hence, this model provides a large number of counterexamples to the above idea. We then study in detail the solutions around infinity which can be extracted from these Kowalevski developments. Using Padé approximants, we are able to extend these solutions beyond the first new singularity and observe how the new singularities obey Kowalevski exponents.

We also comment on the Poisson structure of the model, which is interesting due to the constraints between the dynamical variables, and the Poisson brackets of the variables appearing in the Laurent series, which happens to be of a nice canonical form. We notice that this illustrates the fact that it is the global character of the conserved quantities that is of importance in defining an integrable system.

One of us (MT) is happy to acknowledge useful conversations with J P Ramis[3] and J Sauloy from Toulouse University, about their work on differential Galois theory applied to

[3] http://www.maia.ub.es/dsg/2009/0903simo.ps.gz

the swinging Atwood's machine. Finally we are happy to thank the Maxima team[4] for their software, with which we have performed the computations in this paper.

2. Hamiltonian setup

The description we have given of the swinging Atwood's machine is a constrained system in the Lagrange formulation, so that the equations of motion take a nice algebraic form.

In the articles [1–7] polar coordinates are used, so the constraint is 'solved' but the price to pay is the use of trigonometric functions. Using polar coordinates $x = r\sin\theta$, $y = -r\cos\theta$ the Hamiltonian reads

$$H = \frac{1}{2(m+M)}p_r^2 + \frac{1}{2mr^2}p_\theta^2 + gr(M - m\cos\theta), \tag{6}$$

where $p_r = (m+M)\dot{r}$ and $p_\theta = mr^2\dot{\theta}$.

We now give a Hamiltonian description of this system, using as dynamical variables the three coordinates x, y, z and the three momenta p_x, p_y, p_z with canonical Poisson brackets. The constraint

$$C_1 \equiv z^2 - x^2 - y^2 = 0 \tag{7}$$

generates the flow

$$\{C_1, p_x\} = -2x, \qquad \{C_1, p_y\} = -2y, \qquad \{C_1, p_z\} = 2z \tag{8}$$

which is also generated by the one parameter group acting on phase space by $(x, y, z) \to (x, y, z)$, $(p_x, p_y, p_z) \to (p_x - \mu x, p_y - \mu y, p_z + \mu z)$, where μ is the group parameter.

We want to describe the dynamics of our model as a Hamiltonian system obtained by reduction of an invariant system under this group action [11]. In order to do that, consider the functions

$$\begin{aligned} A_x &= zp_y + yp_z \\ A_y &= zp_x + xp_z \\ A_z &= xp_y - yp_x. \end{aligned}$$

These functions Poisson commute with the constraint C_1 and hence are invariant under the group action. They are not independent however, since they are related by

$$yA_y - xA_x + zA_z = 0. \tag{9}$$

It is easy to check the Poisson brackets

$$\begin{array}{lll} \{A_x, A_y\} = -A_z, & \{A_x, A_z\} = -A_y, & \{A_y, A_z\} = A_x \\ \{A_x, x\} = 0, & \{A_x, y\} = z, & \{A_x, z\} = y \\ \{A_y, x\} = z, & \{A_y, y\} = 0, & \{A_y, z\} = x \\ \{A_z, x\} = -y, & \{A_z, y\} = x, & \{A_z, z\} = 0. \end{array}$$

Let us consider the invariant Hamiltonian

$$H = \frac{1}{2(m+M)z^2}\left[A_x^2 + A_y^2 + \frac{M}{m}A_z^2\right] + Mgz + mgy. \tag{10}$$

To check that H generates the equations of motion on the reduced system, we compute

$$\dot{x} = \{H, x\} = \frac{1}{m+M}\frac{1}{z^2}\left(zA_y - \frac{M}{m}yA_z\right) \tag{11}$$

[4] http://maxima.sourceforge.net/.

$$\dot{y} = \{H, y\} = \frac{1}{m+M}\frac{1}{z^2}\left(zA_x + \frac{M}{m}xA_z\right) \tag{12}$$

$$\dot{z} = \{H, z\} = \frac{1}{m+M}\frac{1}{z^2}(xA_y + yA_x). \tag{13}$$

The right-hand sides of these equations are linear in the momenta p_x, p_y, p_z; however, we cannot invert the system uniquely in order to express the momenta in terms of the velocities. This is because, due to the symmetry ($\{H, C_1\} = 0$) we have $x\dot{x} + y\dot{y} = z\dot{z}$ so the equations are not independent. The solution is

$$p_x = m\dot{x} + \mu x \tag{14}$$

$$p_y = m\dot{y} + \mu y \tag{15}$$

$$p_z = M\dot{z} - \mu z, \tag{16}$$

where μ is arbitrary. Similarly we compute $\ddot{x} = \{H, \dot{x}\}$, etc where $\dot{x}$, etc are the right-hand sides of the above equations. Performing this calculation and using the constraint C_1 and equation (9), we obtain the Lagrangian equation of motion (1)–(3), with λ given by

$$\lambda = \frac{mM}{m+M}\frac{1}{z^2}\left[g(y-z) - \frac{1}{m^2z^2}A_z^2\right].$$

This coincides with equation (5), as can be checked using again equations (14)–(16) and the constraint C_1, to express A_z in terms of $\dot{x}$, $\dot{y}$, $\dot{z}$.

Finally we express the energy in terms of velocities still using the constraints. We find

$$E = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{M}{2}\dot{z}^2 + g(my + Mz), \tag{17}$$

which agrees with what we expect from the Lagrangian formulation.

3. The integrable case

In order to understand what sort of Laurent expansions appear in the model, it is useful to first consider the case $M/m = 3$ which has been integrated by Tufillaro [4]. Let us recall some of his results. He discovered that using polar coordinates (r, θ) such that $x = r\sin\theta$, $y = -r\cos\theta$ and $r = z$, and setting

$$\xi^2 = z[1 + \sin(\theta/2)], \qquad \eta^2 = z[1 - \sin(\theta/2)],$$

the Hamilton–Jacobi equation separates into the variables (ξ, η). These look like parabolic coordinates, except that the half-angle $\theta/2$ is used. Knowing ξ and η, one can recover x and y by

$$x_\pm \equiv x \pm \mathrm{i}y = \pm\frac{\mathrm{i}}{2}\frac{(\xi \mp \mathrm{i}\eta)^3}{(\xi \pm \mathrm{i}\eta)}, \qquad z = \frac{1}{2}(\xi^2 + \eta^2). \tag{18}$$

In fact, just for $M = 3m$, two terms involving couplings between ξ and η disappear, and one gets with momenta $p_\xi = 4\dot{\xi}(\xi^2 + \eta^2)$ etc the expression of the Hamiltonian, in which we have set $m = 1$:

$$H = \left[\left(p_\xi^2 + p_\eta^2\right)/8 + 2g(\xi^4 + \eta^4)\right]/(\xi^2 + \eta^2).$$

Then it is clear that in this case the action S separates as a sum $S_\xi(\xi) + S_\eta(\eta)$ where S_ξ and S_η obey different elliptic equations (corresponding to different elliptic moduli):

$$(\partial_\xi S_\xi)^2 = -16g\xi^4 + 8E\xi^2 + I \equiv P_+(\xi) \tag{19}$$

$$(\partial_\eta S_\eta)^2 = -16g\eta^4 + 8E\eta^2 - I \equiv P_-(\eta), \tag{20}$$

where I is the separation constant. It can be expressed in terms of dynamical variables by subtracting the above two equations multiplied resp. by η^2 and ξ^2, which eliminates E. Moreover, we replace

$$\partial_\xi S = p_\xi = 4\dot{\xi}(\xi^2 + \eta^2), \qquad \partial_\eta S = p_\eta = 4\dot{\eta}(\xi^2 + \eta^2). \tag{21}$$

We get

$$I/16 = (\xi^2 + \eta^2)(\eta^2\dot{\xi}^2 - \xi^2\dot{\eta}^2) + g\xi^2\eta^2(\xi^2 - \eta^2)/(\xi^2 + \eta^2).$$

Returning to polar coordinates, the integral of motion takes the form

$$I/16 = r^2\dot{\theta}\left[\dot{r}\cos(\theta/2) - \frac{r\dot{\theta}}{2}\sin(\theta/2)\right] + gr^2\sin(\theta/2)\cos^2(\theta/2).$$

We want to see if the equations of motion admit a solution which diverges at finite time, and in that case what is the behavior of the Laurent expansion.

The general solution of the Hamilton–Jacobi equation is

$$S = -Et + \int^\xi \sqrt{P_+(\xi)}\,\mathrm{d}\xi + \int^\eta \sqrt{P_-(\eta)}\,\mathrm{d}\eta.$$

According to the general theory, we get the solution of the equations of motion by writing $\partial_E S = c_E$ and $\partial_I S = c_I$ for two constants c_E and c_I. For $c_I \neq 0$, we get

$$t + c_E = \int^\xi \frac{4\xi^2}{\sqrt{P_+(\xi)}}\,\mathrm{d}\xi + \int^\eta \frac{4\eta^2}{\sqrt{P_-(\eta)}}\,\mathrm{d}\eta \tag{22}$$

$$c_I = \frac{1}{2}\int^\xi \frac{1}{\sqrt{P_+(\xi)}}\,\mathrm{d}\xi - \frac{1}{2}\int^\eta \frac{1}{\sqrt{P_-(\eta)}}\,\mathrm{d}\eta. \tag{23}$$

For $I = 0$, the elliptic integrals degenerate to trigonometric ones. We get

$$t + c_E = -\frac{1}{2\omega}\left(\sqrt{1-\alpha\xi^2} + \sqrt{1-\alpha\eta^2}\right), \qquad \alpha = 2g/E, \quad \omega = g/\sqrt{2E} \tag{24}$$

$$\frac{1-\sqrt{1-\alpha\xi^2}}{1+\sqrt{1-\alpha\xi^2}} = K^2\frac{1-\sqrt{1-\alpha\eta^2}}{1+\sqrt{1-\alpha\eta^2}}, \qquad K^2 = e^{c_I}, \tag{25}$$

so that setting $\xi = \sin(\phi_\xi)/\sqrt{\alpha}$, $\eta = \sin(\phi_\eta)/\sqrt{\alpha}$ the second equality reads

$$\tan(\phi_\xi/2) = K\tan(\phi_\eta/2).$$

Using the variable $s = \tan(\phi_\xi/2)$, ξ and η can be expressed rationally:

$$\xi = \frac{1}{\sqrt{\alpha}}\frac{2s}{1+s^2}, \qquad \eta = \frac{1}{\sqrt{\alpha}}\frac{2Ks}{K^2+s^2}$$

Finally, one gets the time variation of $S \equiv s^2$ by using equation (24) which implies

$$\omega\,\mathrm{d}t = \mathrm{d}S\left[\frac{1}{(1+S)^2} + \frac{K^2}{(K^2+S)^2}\right] = -\frac{8\mathrm{i}K}{K^2-1}\frac{U\,\mathrm{d}U}{(U^2-1)^2},$$

where we have parametrized S as

$$S = \mathrm{i}K\frac{(K+\mathrm{i})U + (K-\mathrm{i})}{(K-\mathrm{i})U - (K+\mathrm{i})}.$$

The variable U has been defined to send the poles $S = -K^2$ and $S = -1$ to $U = \pm 1$. One gets the two parameters solution (parameters K and E) up to an origin for time, which we fix by requiring that $t = 0$ for $U = 0$:

$$U^2 = \frac{t}{t - t_\infty} \quad \text{or} \quad U^2 - 1 = \frac{t_\infty}{t - t_\infty} \Longrightarrow t = -t_\infty \frac{U^2}{1 - U^2}, \qquad t_\infty = \frac{1}{\omega} \frac{4\mathrm{i}K}{K^2 - 1}.$$

We will soon see that $t = t_\infty$ is a second singularity of the dynamical variables, that we can express explicitly. For ease of comparison with the following, we present $x_\pm(t) = x(t) \pm \mathrm{i}y(t)$:

$$\begin{aligned}
x_+ &= -\frac{2Kg}{\omega^2} \frac{[(K - \mathrm{i})U - K - \mathrm{i}]\,[(K + \mathrm{i})U + K - \mathrm{i}]}{(K^2 - 1)^2 (U^2 - 1)^2} \frac{1}{U} \\
x_- &= \frac{2Kg}{\omega^2} \frac{[(K - \mathrm{i})U - K - \mathrm{i}]\,[(K + \mathrm{i})U + K - \mathrm{i}]}{(K^2 - 1)^2 (U^2 - 1)^2} U^3 \\
z &= \mathrm{i}\frac{2Kg}{\omega^2} \frac{[(K - \mathrm{i})U - K - \mathrm{i}]\,[(K + \mathrm{i})U + K - \mathrm{i}]}{(K^2 - 1)^2 (U^2 - 1)^2} U \\
\lambda &= -\frac{3\omega^2}{64K^2} \frac{(K^2 - 1)^2 (K^2 + 1)(U^2 - 1)^5}{[(K - \mathrm{i})U - K - \mathrm{i}]\,[(K + \mathrm{i})U + K - \mathrm{i}]\,U^4}.
\end{aligned}$$

In terms of the t variable, we get the simpler expressions

$$x_+(t) = -\frac{2Kg}{\omega^2 (K^2 - 1)^2} \left[(K^2 + 1) \left(\frac{t - t_\infty}{t_\infty} \right)^{3/2} \left(\frac{t_\infty}{t} \right)^{1/2} - 4\mathrm{i}K \left(\frac{t - t_\infty}{t_\infty} \right)^2 \right] \tag{26}$$

$$x_-(t) = \frac{2Kg}{\omega^2 (K^2 - 1)^2} \left[(K^2 + 1) \left(\frac{t}{t_\infty} \right)^{3/2} \left(\frac{t_\infty}{t - t_\infty} \right)^{1/2} - 4\mathrm{i}K \left(\frac{t}{t_\infty} \right)^2 \right]. \tag{27}$$

We see that x_+ behaves as $t^{-\frac{1}{2}}$ and x_- behaves as $t^{\frac{3}{2}}$ when $t \to 0$. If we expand around $t = 0$, we get Puiseux expansions in $t^{\frac{1}{2}}$. These expansions depend on three parameters, K and E plus the origin of time t_0. This is because we are analyzing the trigonometric solution which fixes one of the constants to $I = 0$. We will see later on that it can be generalized to a four-parameter expansion in the elliptic case. The energy parameter appears factorized in front of x_+ and x_- in the form of $g/\omega^2 = 2E/g$.

Around t_∞, we see that x_+ behaves as $(t - t_\infty)^{\frac{3}{2}}$ and x_- behaves as $(t - t_\infty)^{-\frac{1}{2}}$ which is symmetrical with the behavior at $t = 0$. This is compatible with the fact that the equations of motion admit a symmetry $(x_+(t), x_-(t)) \leftrightarrow (-x_-(t), -x_+(t))$.

Remark that $x_\pm(t)$ are defined on the two-sheeted covering of the Riemann sphere with two branch points at $t = 0$ and $t = t_\infty$. The variable U that we have introduced is in fact a uniformizing variable for this covering, so that $x_\pm(t)$ are rational functions of U. Moreover, $U \leftrightarrow -1/U$ corresponds to $t \leftrightarrow (t_\infty - t)$ and exchanges x_+ and $-x_-$. The extra minus sign means that we have to change the determination of the square root in the t variable. The U variable makes this completely unambiguous:

$$x_+\left(-\frac{1}{U}\right) = -x_-(U), \qquad z\left(-\frac{1}{U}\right) = z(U), \qquad \lambda\left(-\frac{1}{U}\right) = \lambda(U).$$

We emphasize that, although the system is integrable, the solutions diverge with *square root* singularities at finite times $t = 0$, and $t = t_\infty$.

We now return to the elliptic case. Let us define the variables $X = \xi^2 - E/(6g)$ and $Y = \eta^2 - E/(6g)$. Equations (22), (23) become

$$t + c_E = \frac{1}{4\mathrm{i}\sqrt{g}} \int^X \frac{(X + E/(6g))\,\mathrm{d}X}{\sqrt{P_+(X)}} + \frac{1}{4\mathrm{i}\sqrt{g}} \int^Y \frac{(Y + E/(6g))\,\mathrm{d}Y}{\sqrt{P_-(Y)}}$$

$$c_I = \frac{1}{4\mathrm{i}\sqrt{g}}\int^X \frac{\mathrm{d}X}{\sqrt{P_+(X)}} - \frac{1}{4\mathrm{i}\sqrt{g}}\int^Y \frac{\mathrm{d}Y}{\sqrt{P_-(Y)}},$$

where now

$$P_\pm(X) = 4X^3 - g_2(\pm I)X - g_3(\pm I)$$
$$g_2(I) = \frac{1}{3g^2}\left(E^2 + \frac{3}{4}gI\right), \qquad g_3(I) = \frac{E}{27g^3}\left(E^2 + \frac{9}{8}gI\right).$$

Introducing the Weierstrass functions

$$X = \wp_1(Z_1) \equiv \wp(Z_1, g_2(I), g_3(I)), \qquad Y = \wp_2(Z_2) \equiv \wp(Z_2, g_2(-I), g_3(-I))$$

the above integrals reduce to

$$t + c_E = \frac{1}{4\mathrm{i}\sqrt{g}}\left[\frac{E}{6g}(Z_1+Z_2) - \zeta_1(Z_1) - \zeta_2(Z_2)\right] \tag{28}$$

$$c_I = \frac{1}{4\mathrm{i}\sqrt{g}}[Z_1 - Z_2], \tag{29}$$

where ζ is the Weierstrass zeta function, $\zeta' = -\wp$. The $\wp$ function has two periods $2\omega_j$, $j = 1, 2$, so that $\wp(z + 2\omega_j) = \wp(z)$, but the zeta function is quasi periodic, $\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j$. Here we have two sets of periods ω_j and η_j according to the function $\wp_1$ or $\wp_2$, which are in fact functions $\omega_j(\pm I)$ and $\eta_j(\pm I)$.

Note that $x_\pm(t)$ have poles and zeros when $\xi \pm \mathrm{i}\eta$ vanish, that is when $\xi^2 + \eta^2 = X + Y + E/(3g) = 0$. Hence, we have to solve

$$E/(3g) + \wp_1(Z_1) + \wp_2(Z_2) = 0 \tag{30}$$

$$Z_1 - Z_2 - 4\mathrm{i}\sqrt{g}c_I = 0. \tag{31}$$

But differentiating equations (28), (29) we find $\delta Z_2 = \delta Z_1$ and

$$\delta t = \frac{1}{4\mathrm{i}\sqrt{g}}\left(\frac{E}{3g} + \wp_1(Z_1) + \wp_2(Z_2)\right)\delta Z_1 + \frac{1}{8\mathrm{i}\sqrt{g}}(\wp_1'(Z_1) + \wp_2'(Z_2))(\delta Z_1)^2 + \cdots.$$

The first term vanishes when $\xi^2 + \eta^2 = 0$; hence, around such a zero $\delta Z_1 \simeq \sqrt{\delta t}$. As a consequence, in view of equation (18), $x_\pm(t)$ behaves as either $\delta t^{-1/2}$ or $\delta t^{3/2}$ at such a point, according to the vanishing of $\xi + \mathrm{i}\eta$ or $\xi - \mathrm{i}\eta$. Note that this is similar to the trigonometric case.

However, finding the pattern of these singularities is messy, because in equations (30), (31) we have two incommensurate lattices of periods for the two Weierstrass functions. Nevertheless, we can easily see that there is an infinite number of singularities. This is because since the two lattices are incommensurate, for any large R and small ϵ, one can choose V in the first lattice and W in the second, such that $|V - W| < \epsilon$ and $|V|, |W| > R$. Starting from a solution Z_1, Z_2 of our equations, we set $Z_1' = Z_1 + V$ and $Z_2' = Z_2 + W$, which still obey equation (30). However, equation (31) is violated at order ϵ. Choose $Z_1'' = Z_1'$, $Z_2'' = Z_1'' - 4\mathrm{i}\sqrt{g}c_I$ and plug this in equation (30). It then gets of order ϵ but this is an equation for the variable Z_1 which has, by complex analyticity, an exact solution close to this approximate solution. Taking larger and larger values for R, one gets an infinite number of solutions. Around each of these solutions, we have Puiseux expansions in the variable $\delta t^{1/2}$.

4. Kowalevski analysis

If the swinging Atwood's machine is an algebraically integrable system, the dynamical variables can be expressed algebraically in terms of a linear motion on some Abelian variety; in particular all variables and time can be complexified at will. We may expect that, for general initial conditions, the dynamical variables will blow out for some (in general complex) value t_0 of the time t. Around this value the dynamical variables should have Laurent behavior; hence, one expects to find Laurent solutions depending on N parameters (initial conditions) if the phase space is of dimension N. In practice one searches for Laurent expansions at $t = 0$ (one fixes $t_0 = 0$) so an admissible Laurent solution should have $N - 1$ parameters, that is three parameters for the example at hand.

The Puiseux solutions we have found in the previous section have the following singularity: x and y blow up but $z \to 0$, hence $x^2 + y^2 \to 0$. This means that the singular solutions are such that the mass m goes to the origin but rotating faster and faster. If we expand x and y in negative powers of t, there must be large cancellations such that $x^2 + y^2 \to 0$. It is much more convenient to factorize $x^2 + y^2$ and have the cancellation between the two factors. Recalling that

$$x_\pm = x \pm \mathrm{i}y,$$

the equations of motion are

$$\begin{aligned} m\ddot{x}_+ &= -\mathrm{i}mg + \lambda x_+ \\ m\ddot{x}_- &= \mathrm{i}mg + \lambda x_- \\ M\ddot{z} &= -Mg - \lambda z \\ z^2 &= x_+ x_-. \end{aligned} \tag{32}$$

The value of λ is a consequence of these equations

$$\lambda = \frac{mM}{M+m}\frac{\dot{z}^2 - \dot{x}_+\dot{x}_- + g(y-z)}{z^2},$$

where $y = -\mathrm{i}(x_+ - x_-)/2$. Let us remark that this system of equations is invariant under $(x_+, x_-) \to (-x_-, -x_+)$; in particular y and λ are invariant. The system is also invariant under a similarity transformation:

$$x_\pm(t) \to \mu^2 x_\pm(t/\mu), \qquad z(t) \to \mu^2 z(t/\mu), \qquad \lambda(t) \to \frac{1}{\mu^2}\lambda(t/\mu).$$

We first analyze equations (32) at the leading order. We thus look for solutions of the form

$$x_+ = a_1 t^p + \cdots, \qquad x_- = b_1 t^q + \cdots,$$

so that equation (32) requires

$$z = c_1\, t^{\frac{p+q}{2}} + \cdots, \qquad c_1^2 = a_1 b_1.$$

At the lowest order we then have

$$\lambda = \frac{mM}{4(M+m)}\frac{a_1 b_1 (p-q)^2 t^{p+q-2} + 4g(y-z)}{a_1 b_1 t^{p+q}}. \tag{33}$$

Clearly equations of motion (32) require that λ behave as $1/t^2$ for solutions blowing out as powers. At first sight there are two ways in which this can happen: when the first term in the numerator is dominant, or when the second term is dominant. We can always choose $p \leqslant q$, up to exchange of x_+ and x_-; hence, $p < 0$ since we want to have at least one dynamical variable diverging. On the other hand $z \to 0$ so q is positive; hence, $y - z = O(t^p)$.

The first term is dominant when $q < 2$, and for $p \neq q$ one has indeed $\lambda \simeq 1/t^2$. When $q = 2$ both terms are of the same order and for $q > 2$ the second term is dominant, so that $\lambda = O(t^{-q})$ which is not allowed. Hence, we have basically only two cases to consider, either $p < 0, q < 2$ in which the integrable case studied above belongs to $(p = -1/2, q = 3/2)$, or the case $-2 < p < 0, q = 2$, which, as we will see, covers more general values of the mass ratio M/m.

4.1. Integrable case

Since $p < 0, q < 2$ we have $p + q - 2 < \left(p, q, \frac{p+q}{2}\right)$, and we can neglect the term $g(y - z)$ at leading order in the expression of λ. We find, for $p \neq q$,

$$\lambda = \frac{mM(p-q)^2}{4(M+m)}\frac{1}{t^2} + \cdots .$$

Similarly the equations of motion for $x_\pm$ give

$$p(p-1) = \frac{M}{4(M+m)}(p-q)^2 = q(q-1) \tag{34}$$

so that $(p - q)(p + q - 1) = 0$, hence, since $p \neq q$, $p < q$ and we have $p + q - 1 = 0$. Since by positivity in equation (34), p and q cannot belong to [0, 1] this implies, together with $p > p + q - 2 = -1$, that

$$-1 < p < 0, \qquad 1 < q < 2.$$

Using $p + q = 1$ the mass ratio takes the form

$$M = -4mpq = m[(p-q)^2 - 1] = m[(2p-1)^2 - 1]$$

and the mass ratio M/m is thus in the interval]0, 8[.

The integrable case corresponds to $M = 3m$, and falls into this analysis with

$$p = -\tfrac{1}{2}, \qquad q = \tfrac{3}{2}.$$

These exponents are exactly those we have found in the exact solution of the elliptic integrable case. There are no other values of p in]−1, 0[compatible with integer values of the mass ratio M/m which could, according to [4], correspond to seemingly integrable behavior. We thus consider, in the following, the integrable case $M/m = 3$.

As noted above the second conserved quantity is given in polar coordinates for $m = 1$, introducing for convenience $H_2 = I\sqrt{2}/8$, by

$$\frac{1}{2\sqrt{2}}H_2 = r^2\dot\theta\frac{\mathrm{d}}{\mathrm{d}t}(r\cos(\theta/2)) + \frac{g}{2}(r\sin\theta)(r\cos(\theta/2)),$$

which reads in cartesian coordinates as

$$H_2 = \frac{1}{\sqrt{z(z-y)}}(x\dot y - y\dot x)\frac{\mathrm{d}}{\mathrm{d}t}(z^2 - zy) + gx\sqrt{z(z-y)}.$$

Taking the square to eliminate the square roots, we get

$$H_2^2 = \frac{1}{z(z-y)}(x\dot y - y\dot x)^2\left(\frac{\mathrm{d}}{\mathrm{d}t}(z^2 - zy)\right)^2 + 2gx(x\dot y - y\dot x)\frac{\mathrm{d}}{\mathrm{d}t}(z^2 - zy) + g^2x^2(z^2 - zy).$$

We can set up an expansion in powers of $\sqrt{t}$:

$$\begin{aligned}
x_+ &= t^{-\frac{1}{2}}\left(a_1 + a_2 t^{\frac{1}{2}} + \cdots\right)\\
x_- &= t^{\frac{3}{2}}\left(b_1 + b_2 t^{\frac{1}{2}} + \cdots\right)\\
z &= t^{\frac{1}{2}}\left(d_1 + d_2 t^{\frac{1}{2}} + \cdots\right)\\
\lambda &= t^{-2}\left(l_1 + l_2 t^{\frac{1}{2}} + \cdots\right).
\end{aligned}$$

We already know that

$$a_1 b_1 = d_1^2, \qquad l_1 = \frac{3m}{4}.$$

Inserting into the equations of motion, we find the recursive system

$$\mathcal{K}(s) \cdot \begin{pmatrix} a_{s+1} \\ b_{s+1} \\ d_{s+1} \\ l_{s+1} \end{pmatrix} = \begin{pmatrix} A_{s+1} \\ B_{s+1} \\ D_{s+1} \\ L_{s+1} \end{pmatrix}$$

$$\mathcal{K}(s) = \begin{pmatrix} m\frac{(s-1)(s-3)}{4} - l_1 & 0 & 0 & -a_1 \\ 0 & m\frac{(s+1)(s+3)}{4} - l_1 & 0 & -b_1 \\ 0 & 0 & M\frac{(s+1)(s-1)}{4} + l_1 & d_1 \\ -b_1 & -a_1 & 2d_1 & 0 \end{pmatrix}.$$

The square matrix in the left-hand side is called the Kowalevski matrix, and the vector in the right-hand side is given by

$$A_{s+1} = \sum_{j=1}^{s-1} l_{j+1} a_{s-j+1} - \mathrm{i}mg\, \delta_{s,5}$$

$$B_{s+1} = \sum_{j=1}^{s-1} l_{j+1} b_{s-j+1} + \mathrm{i}mg\, \delta_{s,1}$$

$$D_{s+1} = -\sum_{j=1}^{s-1} l_{j+1} d_{s-j+1} - Mg\delta_{s,3}$$

$$L_{s+1} = -\sum_{j=1}^{s-1} d_{j+1} d_{s-j+1} + \sum_{j=1}^{s-1} a_{j+1} b_{s-j+1}.$$

The determinant of the Kowalevski matrix reads

$$\det(\mathcal{K}(s)) = -\frac{m^2 d_1^2}{2}(s+2)s^2(s-2).$$

It has a double zero at $s = 0$ and a third zero at the integer value $s = 2$. Hence, potentially three arbitrary constants may appear in the expansion. Indeed the miracle happens at the third level where the equations determining the coefficients a_3, b_3 are degenerate, leaving one extra constant $b_3 = c_1$. The rest of the expansion is then completely determined at all orders. We find in particular (in the following numerical computations, $g = 1$)

$$x_+ = \frac{d_1^2}{b_1\sqrt{t}} + \frac{\mathrm{i}d_1^2 g}{2b_1^2} - \frac{3c_1 d_1^2\sqrt{t}}{b_1^2} + \frac{\left(4\mathrm{i}c_1 d_1^2 - 7b_1^2 d_1\right) gt}{5b_1^3}$$
$$+ \frac{\left(\left(2c_1 d_1^2 + \mathrm{i}b_1^2 d_1\right) g^2 + 12 b_1 c_1^2 d_1^2\right) t^{\frac{3}{2}}}{8b_1^4} + \cdots$$

$$x_- = b_1 t^{\frac{3}{2}} + \frac{\mathrm{i}gt^2}{2} + c_1 t^{\frac{5}{2}} - \frac{\left(2\mathrm{i}c_1 d_1 - b_1^2\right) gt^3}{5b_1 d_1}$$
$$- \frac{\left(\left(6c_1 d_1 + 3\mathrm{i}b_1^2\right) g^2 - 60 b_1 c_1^2 d_1\right) t^{\frac{7}{2}}}{40 b_1^2 d_1} + \cdots.$$

The existence of such a 'miracle' is exactly what Kowalevski noted in [8] for her integrable case of the top. For this to happen one needs that the determinant of $\mathcal{K}(s)$ vanishes for the correct number of *integer* values of the recursive variable s, which allows for a new indeterminate to enter the expansion. Moreover, in this case the linear system has to be solvable which is far from guaranteed. The *general* solution of the equations of motion must admit a power series expansion, which thus must depend on $2N-1$ arbitrary constants for a system of N degrees of freedom. In our case we find a solution depending correctly on three constants, which extends the trigonometric solution described above.

Inserting these expansions into the formula for the energy (17) we obtain

$$E = -\frac{md_1^2}{8b_1^2}(g^2 + 32c_1b_1).$$

Similarly, the second conserved quantity reads

$$H_2^2 = \frac{2\mathrm{i}d_1^5}{b_1^3}\left(b_1^2 - 2\mathrm{i}c_1d_1\right)^2.$$

It is interesting to compare these general results to the expansion in the trigonometric case equations (26), (27). One finds

$$\begin{aligned}
b_1 &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\frac{g(K^2+1)}{4\sqrt{\omega}\sqrt{K}\sqrt{K^2-1}}\\
c_1 &= \mathrm{e}^{-\frac{3\mathrm{i}\pi}{4}}\frac{g\sqrt{\omega}\sqrt{K^2-1}(K^2+1)}{32K^{\frac{3}{2}}}\\
d_1 &= \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}\frac{g\sqrt{K}(K^2+1)}{\omega^{\frac{3}{2}}(K^2-1)^{\frac{3}{2}}}.
\end{aligned}$$

With these values one checks that $H_2 = 0$ as it should be in the trigonometric case, and that H is indeed equal to E.

The dynamical variables (x, y, z) and their time derivatives are expressed in power series of $\sqrt{t}$. These power series have a nonvanishing finite radius of convergence (we know this at least in the trigonometric case from the exact solution) and we can check it numerically. To do that we compute the d'Alembert quotient $|a_{n+1}/a_n|$ relative to a series $\sum_n a_n t^n$ which tends to the inverse of the radius of convergence of this series when it exists. We present the result of this computation for high order n for the series $x_+(t)$, $x_-(t)$, $z(t)$ and $\lambda(t)$ in figure 2.

In this and following similar computations, all values are calculated with absolute precision rational numbers using a formal computation tool. This ensures accuracy of the result.

Since the Kowalevski expansion converges in a disk, the parameters (b_1, c_1, d_1) appearing in these series, and the origin of time t_0, can be considered as coordinates on an open set of phase space near infinity [12]. The question then arises to compute the Poisson brackets in these coordinates.

To do that, we start from

$$\{A_z(t), x_\pm(t)\} = \pm\mathrm{i}x_\pm(t). \tag{35}$$

This equation is valid for any time since the time evolution is a canonical transformation. We thus insert into it the series for $x_\pm(t)$, where these series are really series in $(t+t_0)^{\frac{1}{2}}$. Similarly

$$A_z(t) = \mathrm{i}\frac{m}{2}(x_+\dot{x}_- - x_-\dot{x}_+)(t)$$

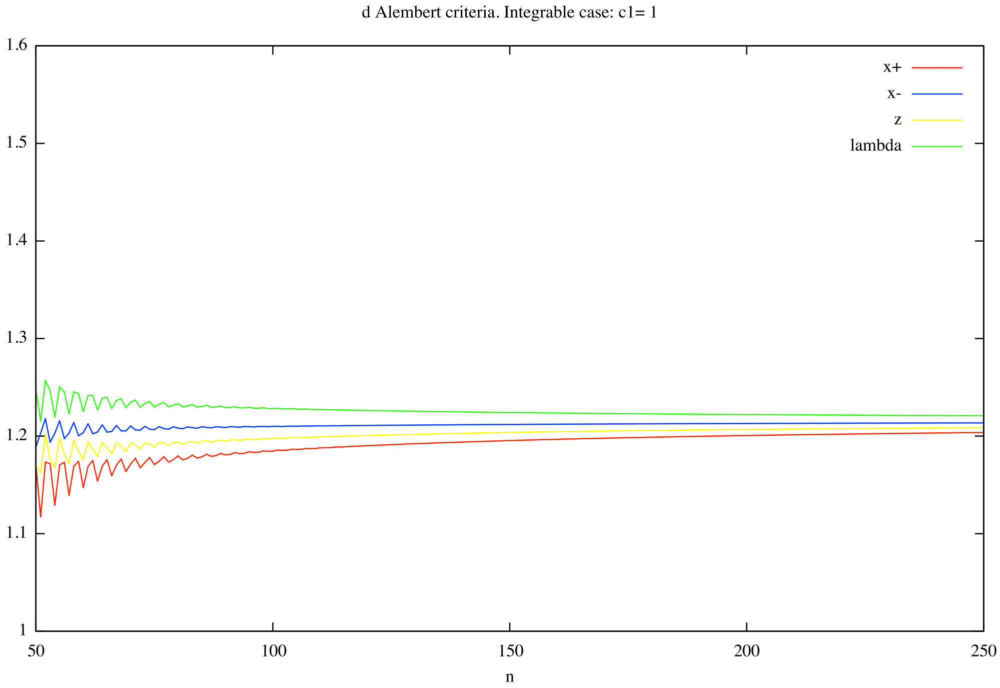

Figure 2. d'Alembert criterion for convergence for $p = -1/2, q = 3/2$. $b_1 = c_1 = d_1 = 1$.

is expressed as a series in $(t + t_0)^{\frac{1}{2}}$ and equation (35) is an identity in t. The Poisson bracket is computed with the rule

$$\begin{aligned}\{F, G\} &= \left(\frac{\partial F}{\partial t_0}\frac{\partial G}{\partial b_1} - \frac{\partial G}{\partial t_0}\frac{\partial F}{\partial b_1}\right)\{t_0, b_1\} + \left(\frac{\partial F}{\partial t_0}\frac{\partial G}{\partial c_1} - \frac{\partial G}{\partial t_0}\frac{\partial F}{\partial c_1}\right)\{t_0, c_1\}\\ &\quad + \left(\frac{\partial F}{\partial t_0}\frac{\partial G}{\partial d_1} - \frac{\partial G}{\partial t_0}\frac{\partial F}{d_1}\right)\{t_0, d_1\} + \left(\frac{\partial F}{\partial b_1}\frac{\partial G}{\partial c_1} - \frac{\partial G}{\partial b_1}\frac{\partial F}{\partial c_1}\right)\{b_1, c_1\}\\ &\quad + \left(\frac{\partial F}{\partial b_1}\frac{\partial G}{\partial d_1} - \frac{\partial G}{\partial b_1}\frac{\partial F}{\partial d_1}\right)\{b_1, d_1\} + \left(\frac{\partial F}{\partial c_1}\frac{\partial G}{\partial d_1} - \frac{\partial G}{\partial c_1}\frac{\partial F}{\partial d_1}\right)\{c_1, d_1\}.\end{aligned}$$

Plugging $F = A_z(t + t_0)$ and $G = x_\pm(t + t_0)$ and identifying term by term in $(t + t_0)$ we get an infinite system for the six Poisson brackets of the coordinates, which is compatible, and whose solution is given by

$$\begin{aligned}\{t_0, d_1\} &= 0\\ \{t_0, b_1\} &= 0\\ \{t_0, c_1\} &= \frac{b_1}{4md_1^2}\\ \{b_1, d_1\} &= \frac{b_1}{2md_1}\\ \{c_1, d_1\} &= \frac{g^2 + 16b_1c_1}{32mb_1d_1}\\ \{c_1, b_1\} &= \frac{g^2 + 32b_1c_1}{32md_1^2}.\end{aligned}$$

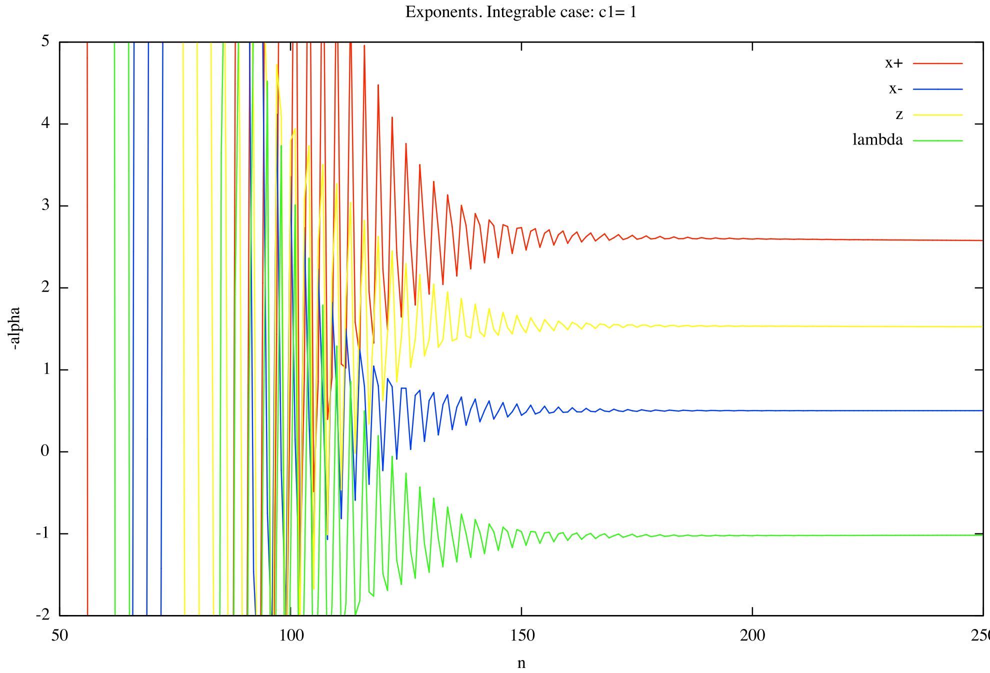

Figure 3. Exponents $-\alpha$ as functions of n, $p = -1/2$, $q = 3/2$. $b_1 = c_1 = d_1 = 1$.

We can then check that

$$\{H, b_1\} = \{H, c_1\} = \{H, d_1\} = 0, \qquad \{H, t_0\} = 1.$$

Finally we see that canonical coordinates can be chosen to be the pair of couples (H, t_0) and $\left(\log b_1, md_1^2\right)$; hence, the Kowalevski constants are essentially Darboux coordinates in a neighborhood of infinity.

This shows the interest of these Darboux coordinates in a vicinity of infinity, but the whole question of integrability is a global one. Our problem is therefore to try to extract some information from the Kowalevski series beyond their disk of convergence. In the following, we investigate this problem numerically. First, we have seen that a_{n+1}/a_n tends to a complex number that we call with hindsight $t_\infty^{-1/2}$. Hence, a_n behaves asymptotically as $a_n \simeq t_\infty^{-n/2}$. One can do even better and look at the prefactor. Assuming that

$$a_n \simeq A n^\alpha t_\infty^{-n/2},$$

we can extract the coefficient α by computing the quantity

$$\lim_{n\to\infty} n^2 \left[\frac{a_{n-2}a_n}{a_{n-1}^2} - 1 \right] = -\alpha.$$

We show the result of this calculation in figure 3.

Note that the curves begin by large oscillations but for n sufficiently large, in the asymptotic regime, the exponents α tend to constants. Comparing with the dominant terms in the binomial formula

$$\sum n^\alpha z^n \simeq_{z\to 1} (1 - z)^{-1-\alpha},$$

we see that setting $z = \sqrt{t/t_\infty}$, we read from figure 3 the various exponents:

$$x_+(t) \simeq (1-z)^{3/2},$$
$$x_-(t) \simeq (1-z)^{-1/2},$$
$$z(t) \simeq (1-z)^{1/2},$$
$$\lambda(t) \simeq (1-z)^{-2}.$$

The consequence of this observation is that $x_\pm(t)$ have Kowalevski expansions around t_∞ with indices which are exchanged as compared to those around $t = 0$. Hence, we know that

$$x_+ = -b_1'(t_\infty - t)^{\frac{3}{2}} - \frac{\mathrm{i}g(t_\infty - t)^2}{2} - c_1'(t_\infty - t)^{\frac{5}{2}} + \cdots$$
$$x_- = -\frac{d_1'^2}{b_1'\sqrt{t_\infty - t}} - \frac{\mathrm{i}d_1'^2 g}{2b_1'^2} + \frac{3c_1'd_1'^2\sqrt{t_\infty - t}}{b_1'^2} + \cdots,$$

where we have introduced a change of sign required by the symmetry $x_\pm \to -x_\mp$, and the symmetry of the equations of motion under $t \to t_\infty - t$. The series expansions have new parameters b_1', c_1' and d_1'. In the trigonometric case we see from the explicit formulae that they are equal to the original parameters, see equations (26), (27).

We have learned from the previous analysis that the singularities are always of the Kowalevski type, with well-defined exponents. This is perfectly consistent with the exact solution in the trigonometric and elliptic case.

4.2. Nonintegrable case

We now explore the region of parameters $-2 < p < 0, q = 2$. We assume that

$$x_+ \simeq a_1 t^p, \qquad x_- \simeq b_1 t^2, \qquad z \simeq c_1 t^{\frac{p}{2}+1}, \qquad c_1^2 = a_1 b_1.$$

Note that $z \to 0$ since we assume $p > -2$, and that $y = -\frac{\mathrm{i}}{2}(x_+ - x_-) \simeq -\frac{\mathrm{i}}{2}a_1 t^p$. We see that both terms in equation (33) for λ contribute

$$\lambda \simeq \frac{mM}{M+m}\left(\left(\frac{p}{2}-1\right)^2 - \frac{\mathrm{i}g}{2b_1}\right)\frac{1}{t^2}.$$

The $x_\pm$ equation give

$$mp(p-1) = \frac{mM}{M+m}\left(\left(\frac{p}{2}-1\right)^2 - \frac{\mathrm{i}g}{2b_1}\right)$$
$$2mb_1 = \mathrm{i}mg + \frac{mM}{M+m}\left(\left(\frac{p}{2}-1\right)^2 - \frac{\mathrm{i}g}{2b_1}\right)b_1.$$

Solving for b_1 we find

$$M = -4m\frac{p-1}{p+2}, \qquad b_1 = -\frac{\mathrm{i}g}{(p-2)(p+1)}.$$

Note that the mass ratio is positive if $-2 < p < 0$, and that

$$\lambda \simeq \frac{mp(p-1)}{t^2}.$$

For relatively prime integers r and k we set

$$p = -\frac{r}{k}, \qquad -2k < -r < -k.$$

We perform the Puiseux expansions

$$x_+ = t^{-\frac{r}{k}}\left(a_1 + a_2 t^{\frac{1}{k}} + \cdots\right)$$
$$x_- = t^2\left(b_1 + b_2 t^{\frac{1}{k}} + \cdots\right)$$
$$z = t^{-\frac{r}{2k}+1}\left(d_1 + d_2 t^{\frac{1}{k}} + \cdots\right)$$
$$\lambda = t^{-2}\left(l_1 + l_2 t^{\frac{1}{k}} + \cdots\right).$$

We already know that

$$l_1 = m\frac{r(r+k)}{k^2}, \qquad a_1 = \frac{d_1^2}{b_1}, \qquad b_1 = -\frac{\mathrm{i}gk^2}{(r+2k)(r-k)}, \qquad M = 4m\frac{r+k}{2k-r}.$$

When we plug this into the equations of motion, we get a system of the form

$$\mathcal{E}_s:\quad \mathcal{K}(s)\cdot\begin{pmatrix} a_{s+1}\\ b_{s+1}\\ d_{s+1}\\ l_{s+1}\end{pmatrix} = \begin{pmatrix} A_{s+1}\\ B_{s+1}\\ D_{s+1}\\ L_{s+1}\end{pmatrix}, \tag{36}$$

where the Kowalevski matrix reads

$$\mathcal{K}(s) = \begin{pmatrix} m\frac{(s-r)(s-r-k)}{k^2} - l_1 & 0 & 0 & -a_1\\ 0 & m\frac{(2k+s)(k+s)}{k^2} - l_1 & 0 & -b_1\\ 0 & 0 & M\frac{(k+s-r/2)(s-r/2)}{k^2} + l_1 & d_1\\ -b_1 & -a_1 & 2d_1 & 0\end{pmatrix}$$

and the right-hand side of equation $\mathcal{E}_s$ is given by

$$\begin{aligned} A_{s+1} &= \sum_{j=2}^{s} a_j l_{s+2-j} - \mathrm{img}\,\delta_{s,2k+r}\\ B_{s+1} &= \sum_{j=2}^{s} b_j l_{s+2-j}\\ D_{s+1} &= -\sum_{j=2}^{s} d_j l_{s+2-j} - Mg\delta_{s,k+r/2}\\ L_{s+1} &= -\sum_{j=2}^{s} d_j d_{s+2-j} + \sum_{j=2}^{s} a_j b_{s+2-j}. \end{aligned} \tag{37}$$

For $s = 1$, the quantities A_2, B_2, D_2, L_2 are meant to be zero. The determinant of the Kowalevski matrix reads

$$\det(\mathcal{K}(s)) = -6m^2 d_1^2\frac{(2k+r)}{k^4(2k-r)}s(s+k)(s-r)(s+k-r).$$

In order that this determinant vanishes for two positive integer values of s, assuming $k > 0$, we should have

$$r > 0, \qquad r - k > 0.$$

From the equation for D_{s+1} it is natural to choose r even; otherwise the weight Mg would disappear from the problem which is not physical. In order for r/k to be irreducible, we must choose k odd. Setting $r = 2r'$ we finally get

$$\frac{k}{2} < r' < k, \qquad p = -2\frac{r'}{k} \tag{38}$$

When things are set up this way the Kowalevski determinant has two strictly positive integer roots, so that, potentially three arbitrary constants enter the expansion, or the expansion is impossible. Impossibility occurs when the right-hand side of equation $\mathcal{E}_s$ is nonvanishing and does not belong to the image of $\mathcal{K}(s)$ for values of s which are Kowalevski indices. It turns out that in most cases the right-hand side vanishes as we now show.

First, since we want to examine the behavior for $s = r-k$ and $s = r$, we can limit ourselves to studying the system for $s = 1, \ldots, r$. In this case the Kronecker deltas in equation (37) always vanish. For $\delta_{s,2k+r}$ it is obvious, and for $\delta_{s,k+r/2}$ note that, since $k \geqslant 1 + r/2$, we have $k + r/2 \geqslant r + 1$. Since the induction starts with $A_2 = B_2 = D_2 = L_2 = 0$, we get, if $s = 1$ is not a Kowalevski index, that $a_2 = b_2 = d_2 = l_2 = 0$; hence, the right-hand side vanishes for the next equation $s = 2$. This goes all the way up to $s = r - k$; hence, when we hit the first Kowalevski index, it is always with vanishing right-hand side. The existence of a nontrivial solution $a_{r-k+1}, \ldots, l_{r-k+1}$ is thus guaranteed. Let us assume for the time being that the first Kowalevski index is such that $(r - k) > 1$, that is $r \geqslant k + 2$.

As a consequence of this previous step, when $s = r - k + 1$ we find that A_{s+1} reduces to $a_{r+k-1}l_2$ which also vanishes because $l_2 = 0$. More generally we have $A_{s+1} = \sum_{j=r-k+1}^{s} a_j l_{s+2-j}$ which vanishes when $s + 2 - j < r - k + 1$ for all j in the sum, and similarly for the other components. This occurs when $s < 2(r - k)$. For $s = 2(r - k)$ the right-hand side of equation $\mathcal{E}_s$ does not vanish, and assuming that we are not on a Kowalevski index, there is a unique nonvanishing solution $a_{2(r-k)+1}, \ldots, l_{2(r-k)+1}$. The process continues and it is easy to show by induction that the right-hand side of equation $\mathcal{E}_s$ does not vanish only for $s = n(r - k)$, n being a positive integer, so nontrivial solutions are of the form $a_{n(r-k)+1}, \ldots, l_{n(r-k)+1}$. Indeed, to get a nonvanishing $a_j l_{s+2-j}$ we need to have $j = n(r-k)+1$ and $s+2-j = n'(r-k)+1$ so that $s = (n+n')(r-k)$. In this case we have only $A_{s+1}, \ldots, L_{s+1}$ and thus $a_{s+1}, \cdots, l_{s+1}$, nonvanishing. This shows that the next nonvanishing positions are of the form $(n + n')(r - k) + 1$ establishing the recurrence.

The second Kowalevski index is $s = r$ and this cannot be of the form $n(r - k)$. Indeed, since r and k are relatively prime, if we have $r = n(r - k)$ we get $(n - 1)r = nk$; hence, $n = pr$ and $n - 1 = qk$ for some integers p and q. Then $(n - 1)r = nk = qkr = prk$ so that $p = q$ and finally $1 = p(r - k)$ which is only possible for $p = 1$ and $r = k + 1$. This is precisely the case we have excluded up to now. As a consequence, when we arrive at the second Kowalevski index $s = r$, the right-hand side of equation $\mathcal{E}_s$ vanishes and there is a nontrivial solution, with an extra constant.

We have shown that two new constants of motion always appear for all cases $r = k + 3, k + 5, \ldots, r = 2k - 2$. This covers an infinite number of values of the mass ratio M/m for which the Kowalevski criterion is satisfied (with weak Painlevé solutions), but for which the system is presumably nonintegrable.

Finally, we discuss the case $k = r + 1$. The first Kowalevski index is $s = 1$. In this case the right-hand side vanishes and we have automatically a nontrivial solution $[a_2, b_2, d_2, l_2]$. From this point, all other solutions of the linear system do not vanish, and in particular, for the second Kowalevski index, $s = r$, the right-hand side of the system is not trivial. For a solution to exist it must be in the image of $\mathcal{K}(r)$. Equivalently, let us consider a covector $U = [u_1, u_2, u_3, u_4]$ such that $U \cdot \mathcal{K}(r) = 0$. Explicitly

$$u_1 = 2g^2k^5, \qquad u_2 = d_1^2(k+1)(3k+1)^2,$$

$$u_3 = \mathrm{i}d_1gk^2(k-1)(3k+1) \qquad u_4 = -2\,\mathrm{i}mg\,k(k+1)(2k+1)(3k+1).$$

The condition to be satisfied is that the scalar product

$$W(s) = u_1A_{s+1} + u_2B_{s+1} + u_3D_{s+1} + u_4L_{s+1}$$

of this covector and the right-hand side of equation (36) vanishes for $s = r = k + 1$.

For arbitrary k and $s = 3, 4, \ldots$ we have computed this scalar product $W(s)$, and we have observed that $W(s)$ has a factor $(s - k + 1)$. For example we get

$$W(s = 3) = -\frac{mc_1^3 d_1^2 (k-2)(k+1)(2k+1)(3k+1)^4}{4g^2 k^5 (k+2)}.$$

Note that the factor $(k - 2) = (r - s)$. For $s = 4$ we next get

$$W(s = 4) = \frac{\mathrm{i} m c_1^4 d_1^2 (k-3)(k+1)(2k+1)(3k+1)^4 P_6(k)}{96 g^3 (k-1)^2 k^8 (k+2)^2 (k+3)}$$

with the factor $(k-3) = (r-s)$. Here c_1 is the Kowalevski constant which has been introduced at $s = 1$, and $P_6(k)$ is some polynomial in k of degree six. The factors in the denominator of course come from similar factors in $\det(\mathcal{K}(s))$. The expression for $s = 5$ has the same type of factors in the numerator and denominator, with a more complicated polynomial $P_7(k)$ and always a factor $(r-s)$. This behavior is persistent as far as one can compute. The consequence of the presence of the factor $(r - s)$ is that, for any k, when we arrive at the second Kowalevski index, $s = r = k + 1$, the scalar product $W(k + 1)$ vanishes and the linear system is solvable. We can thus state that for all admissible pairs (k, r) the swinging Atwood machine has weak Painlevé expansions depending on the full set of parameters.

For example an interesting case occurs when the mass ratio $M/m = 15$ where the system does not look chaotic, see [2]. This case is obtained when $k = 19$ and $r = 26$. The linear system is solvable in this case, although the new arbitrary constants occur very far from the beginning of the expansion. We will refrain to exhibit the solution in this case, since it is very bulky, and proceed to show what happens with smaller values of k and r.

4.3. Example: the case $k = 3, r = 4$

When $k = 3$, we have necessarily $r = 4$. The Kowalevski exponents are $s = 0, s = 1, s = 4$. The dynamical variables $x_\pm$ expand in Puiseux series of $t^{1/3}$ which take the form

$$\begin{aligned} x_+ &= t^{-\frac{4}{3}}\, d_1^2 \left(\frac{10\mathrm{i}}{9g} + 0\, t^{\frac{1}{3}} + \frac{140\mathrm{i}c_1^2}{729g^3} t^{\frac{2}{3}} + \frac{14\,000 c_1^3}{59\,049 g^4} t^{\frac{3}{3}} \right. \\ &\quad \left. + \frac{1960\mathrm{i}c_1^4 m - 32\,805\mathrm{i}c_2 g^4}{91\,854 g^5 m} t^{\frac{4}{3}} + \cdots \right) \\ x_- &= t^2 \left(-\frac{9\mathrm{i}g}{10} + c_1 t^{\frac{1}{3}} + \frac{7ic_1^2}{30g} t^{\frac{2}{3}} + \frac{14c_1^3}{243g^2} t^{\frac{3}{3}} \right. \\ &\quad \left. + \frac{\left(96\,124\mathrm{i}c_1^4 m - 177\,147\mathrm{i}c_2 g^4\right)}{918\,540 g^3 m} t^{\frac{4}{3}} + \cdots \right). \end{aligned}$$

This solution depends on four arbitrary constants: t_0, d_1, c_1, c_2 (in the above expansions t should always be understood as $t + t_0$). We obtain

$$E \equiv H = \frac{5\, d_1^2\, \left(13\,412\, c_1^4\, m - 19\,683\, c_2\, g^4\right)}{91\,854\, g^4}. \tag{39}$$

The above constants can be used as local coordinates on phase space. To compute the Poisson brackets of the Kowalevski constants, we proceed as in the previous section considering $\{A_z(t), x_\pm(t)\} = \pm \mathrm{i} x_\pm(t)$. We find

$$\{t_0, c_1\} = 0 \tag{40}$$

$$\{t_0, d_1\} = 0 \tag{41}$$

$$\{t_0, c_2\} = \frac{14}{15}\,\frac{1}{d_1^2} \tag{42}$$

$$\{d_1, c_1\} = \mathrm{i}\,\frac{3\,g}{20\,m}\,\frac{1}{d_1} \tag{43}$$

$$\{d_1, c_2\} = \mathrm{i}\,\frac{13\,412}{32\,805 g^3}\,\frac{c_1^3}{d_1} \tag{44}$$

$$\{c_1, c_2\} = -\mathrm{i}\,\frac{\left(13\,412\,c_1^4\,m - 19\,683\,c_2\,g^4\right)}{65\,610\,d_1^2\,g^3\,m} = -\mathrm{i}\frac{7g}{25m}\,\frac{E}{d_1^4}. \tag{45}$$

It is remarkable that these six relations ensure the compatibility of an infinite set of relations. One verifies easily the Jacobi identity in spite of the crazy numbers appearing. We can compute the Poisson brackets with H

$$\{H, t_0\} = 1$$
$$\{H, d_1\} = 0$$
$$\{H, c_1\} = 0$$
$$\{H, c_2\} = 0,$$

so that t_0 is the conjugate variable of H as it should be and the other ones are constants of motion. Note that $\left(d_1^2, c_1\right)$ is a pair of canonical variables commuting with the pair (H, t_0). Kowalevski constants are essentially Darboux coordinates.

If there were an extra conserved quantity, it would therefore be a function $F(c_1, c_2, d_1)$. The variable c_2 can be eliminated through H, so that we can write as well $F(H, c_1, d_1)$.

As in the integrable case we can compute numerically the radius of convergence, see figure 4, and the exponents which nicely fit with the above Kowalevski analysis, as shown in figure 5.

To go further, we also compute the Padé approximants of the series. It is more convenient to consider the logarithmic derivatives $\dot{x}_\pm/x_\pm$ because the residues of the poles are the exponents. We present the polar decomposition of the [74, 75] Padé approximant fo $\dot{x}_+/x_+$. This shows clearly, see figure 6, eight true singularities with residues respectively -1.33 and 2 (up to numerical errors) consistent with the Kowalevski analysis. The other poles having small residues correspond to strings of poles and zeros representing algebraic branch cuts in the Padé analysis. Note that we have set $t = z^3$ and we have cancelled the leading z^{-4} at the origin:

$$\begin{aligned}
\dot{x}_+/x_+ = {} & \frac{2.07 + 0.0366\mathrm{i}}{0.812 + 0.618\mathrm{i} + z} + \frac{0.0295 + 0.016\mathrm{i}}{0.813 + 0.622\mathrm{i} + z} + \cdots \\
& + \frac{2.05 - 0.0136\mathrm{i}}{-0.33 + 1.04\mathrm{i} + z} + \frac{0.0351 - 0.007\,92\mathrm{i}}{-0.332 + 1.04\mathrm{i} + z} + \cdots \\
& + \frac{2.13 - 0.0627\mathrm{i}}{-0.95 - 0.012\mathrm{i} + z} + \frac{0.0725 - 0.0176\mathrm{i}}{-0.954 - 1.33 \times 10^{-2}\mathrm{i} + z} + \cdots \\
& + \frac{-1.34 - 0.001\,54\mathrm{i}}{-0.637 - 0.83 \times \mathrm{i} + z} + \frac{-0.007\,11 + 0.0025\mathrm{i}}{-6.49 \times 10^{-1} - 8.52 \times 10^{-1}\mathrm{i} + z} + \cdots \\
& + \frac{2.38 - 0.027\mathrm{i}}{-0.192 - 0.703\mathrm{i} + z} + \frac{0.0281 + 0.0125\mathrm{i}}{-0.192 - 0.705\mathrm{i} + z} + \cdots \\
& + \frac{-1.34 + 3.547 \times 10^{-4}\mathrm{i}}{0.175 - 0.84\mathrm{i} + z} + \frac{-0.0064 - 3.344 \times 10^{-4}\mathrm{i}}{0.177 - 0.85\mathrm{i} + z} + \cdots
\end{aligned}$$

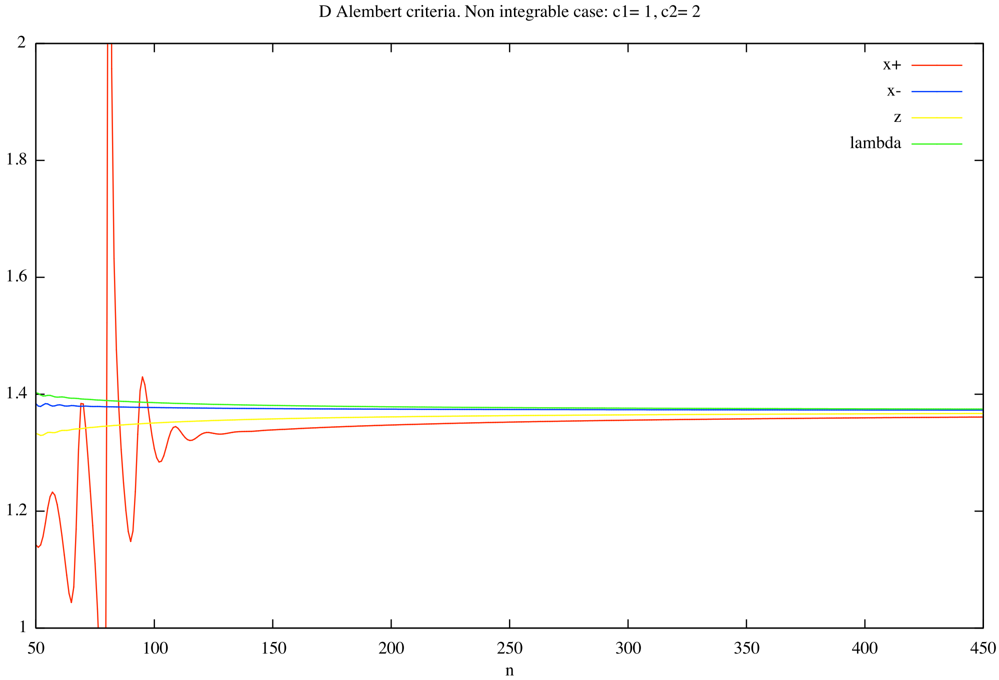

Figure 4. d'Alembert criterion for convergence, $\lim a_{n+1}/a_n$ in the nonintegrable case $c_1 = d_1 = 1, c_2 = 2$ for $N = 450$.

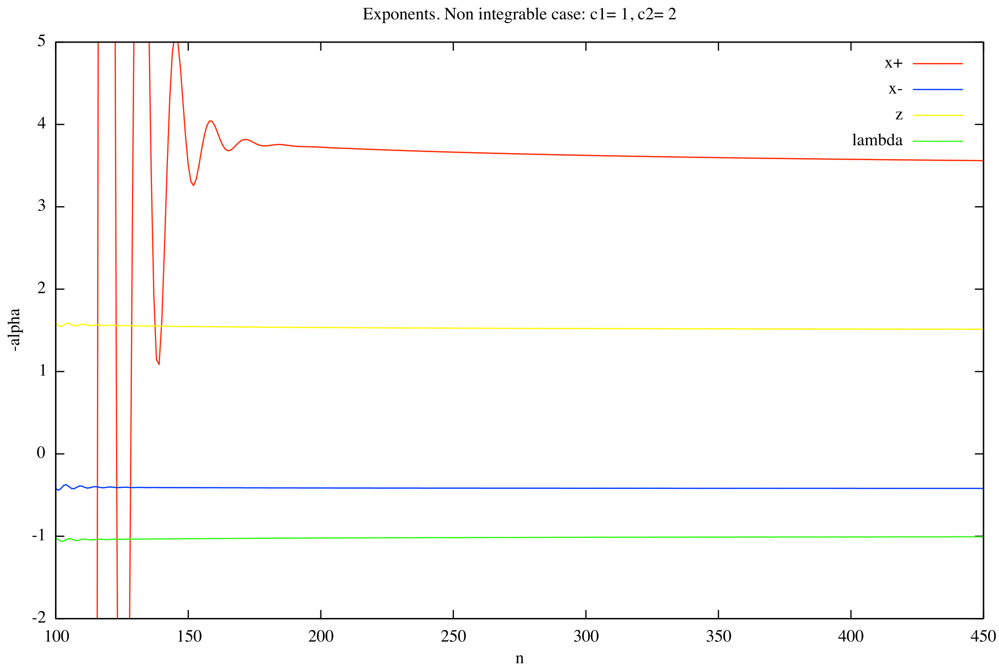

Figure 5. Exponents at singularities in the nonintegrable case $c_1 = d_1 = 1, c_2 = 2$ for $N = 450$.

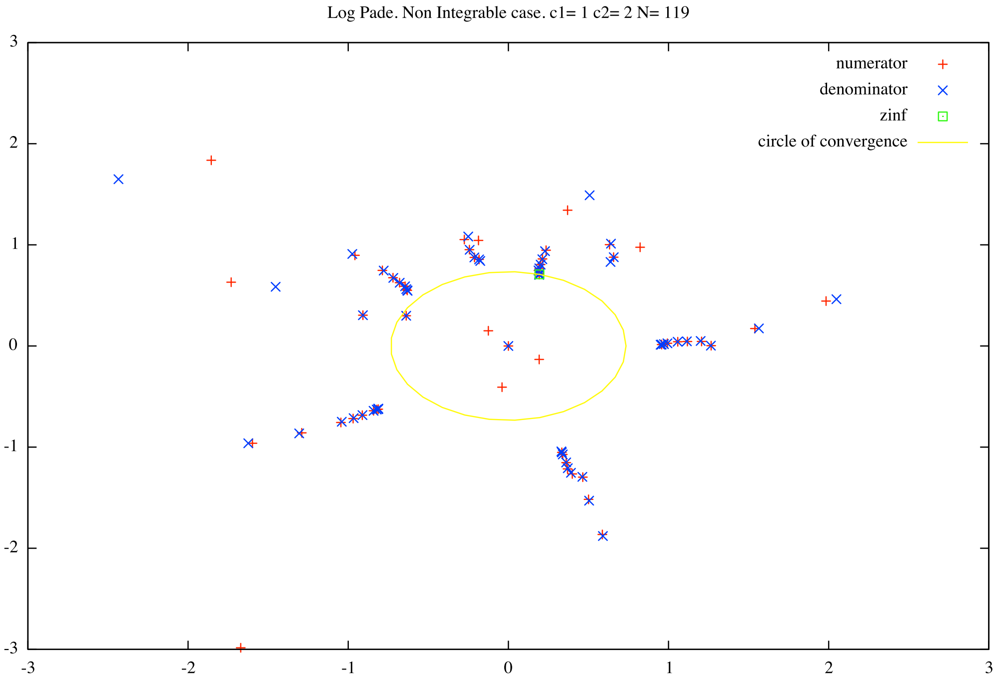

Figure 6. Poles and zeros of Padé approximant $[M, M+1]$ of $\dot{x}_+/x_+$ in the nonintegrable case $c_1 = d_1 = 1$, $c_2 = 2$, $M = 59$ and $N = 119$.

$$
\begin{aligned}
&+ \frac{2.29 + 0.114\mathrm{i}}{0.629 - 0.545\mathrm{i} + z} + \frac{0.077 - 0.0335\mathrm{i}}{0.63 - 0.547\mathrm{i} + z} + \cdots \\
&+ \frac{-1.34 - 0.004\,01\mathrm{i}}{1.45 - 0.586 \times 10^{-1}\mathrm{i} + z} + \frac{0.0802 - 0.0866\mathrm{i}}{1.62 - 0.77\mathrm{i} + z} + \cdots .
\end{aligned}
$$

We see that this structure is very similar to the one we have observed in the integrable elliptic case. This semi-local analysis does not appear to be able to discriminate between the integrable and nonintegrable cases.

5. Conclusion

We have studied the swinging Atwood machine, which is believed to be nonintegrable except for the mass ratio $M/m = 3$. We have shown on the explicit solution of the integrable case that the Kowalevski analysis is valid, but requires weak Painlevé expansions. We have extended this weak Painlevé analysis for other values of the mass ratio, and shown that it is valid for an infinite number of cases. Hence, this model is remarkable in that it exhibits an infinite number of cases where the Kowalevski analysis works at the price of using Puiseux expansions. However, only one of these cases is known to be integrable, while the other ones are believed to be not integrable.

In the cases where Kowalevski expansions are available, we have shown that the constants appearing in these expansions provide Darboux coordinates on an open set of phase space around infinity. The question of integrability of the system therefore reduces to the global nature of this coordinate system (t_0, c_1, c_2, d_1) on phase space.

On this open set, knowing the Poisson brackets equations (40)–(45), we can try to find the conjugate variable of t_0. We find that H must be of the form

$$H = -\tfrac{15}{14} d_1^2 c_2 + h(c_1, d_1).$$

The first term agrees with the exact formula in equation (39). The function $h(c_1, d_1)$ is not determined, but it is of course crucial to have a 'good' function $H(\dot{x}_+, \dot{x}_-, x_+, x_-)$. Clearly we can, in principle, invert locally the system of equations

$$\begin{aligned}
x_+ &= x_+(t - t_0, c_1, c_2, d_1)\\
x_- &= x_-(t - t_0, c_1, c_2, d_1)\\
\dot{x}_+ &= \dot{x}_+(t - t_0, c_1, c_2, d_1)\\
\dot{x}_- &= \dot{x}_-(t - t_0, c_1, c_2, d_1),
\end{aligned}$$

where in the right-hand sides we mean the Kowalevski series. In doing so, we will find

$$\begin{aligned}
t - t_0 &= T(x_+, x_-, \dot{x}_+, \dot{x}_-) \qquad & c_1 &= C_1(x_+, x_-, \dot{x}_+, \dot{x}_-)\\
c_2 &= C_2(x_+, x_-, \dot{x}_+, \dot{x}_-) \qquad & d_1 &= D_1(x_+, x_-, \dot{x}_+, \dot{x}_-)
\end{aligned}$$

but the functions T, C_1, C_2, D_1 will behave in general extremely bad. All this shows that it is in general impossible to make statements about the integrability of the system on the only basis of the Kowalevski analysis. In this context it is remarkable that the global Hamiltonian indeed exists, and it is even more remarkable that a second global Hamiltonian exists in the integrable case. We see here in a striking way the global nature of integrability.

In the nonintegrable case, in an attempt to progress beyond the analysis of a single singularity, we have used Padé expansions. In this semi-local analysis, the panorama which appears is still remarkably similar to the one appearing in the elliptic integrable case. Hence, Kowalevski analysis is not sufficient to characterize integrability. Nevertheless, it is a very nontrivial property whose significance remains mysterious.

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